and

$$e^{-x} = (4/3)^{3/2}u_0(1 - u_0^2/4)^{3/2}$$

An expansion scheme of this type has interesting consequences when introduced into Eqs. (2). For the region of interest ($r \to \infty$) it transpires that the $0(A^{-1})$ term in the Chapman-Enskog expansion does not produce a nonuniformity. However, making an estimate of the error terms in the inviscid expansion using the Navier-Stokes equations, it may be shown for $r \to \infty$ that

$$f_0(T,n,u) = f_0(T_0,n_0,u_0)\{1 + A^{-1}[G \log r + 0(1)] + 0(A^{-2})\} \qquad (5)$$

where G is $0(1)$ as $r \to \infty$. Here the function $f_0(T_0,n_0,u_0)$ is just the Maxwellian distribution corresponding to the inviscid solution. It is now apparent that although the Chapman-Enskog solution $f_0(T,n,u)$ is uniformly valid, the inviscid solution $f_0(T_0,n_0,u_0)$ is not due to the logarithmic singularity at infinity occurring in Eq. (5). The purpose of the following work is to determine suitable functions $\tau_0(r,A)$, $\rho_0(r,A)$, and $V_0(r,A)$ so that $f_0(\tau_0,\rho_0,V_0)$ is a uniformly valid approximation to f as $r \to \infty$.

We return to the Navier-Stokes equations and slightly strain the coordinate x. We write

$$x = s + A^{-1}X_1(s) + A^{-2}X_2(s) + \ldots \qquad (6)$$

together with the expansions

$$\begin{aligned} T(x,A) &= \tau_0(s) + A^{-1}\tau_1(s) + \ldots \\ u(x,A) &= V_0(s) + A^{-1}V_1(s) + \ldots \end{aligned} \qquad (7)$$

Inserting these into Eqs. (3) we have the zeroth order in A^{-1}

$$\begin{aligned} V_0^2\frac{dV_0}{ds} + \frac{2}{5}\left\{V_0\frac{d\tau_0}{ds} - \tau_0\frac{dV_0}{ds} - V_0\tau_0\right\} &= 0 \\ V_0\frac{d\tau_0}{ds} + \frac{2}{3}\left\{\tau_0V_0 + \tau_0\frac{dV_0}{ds}\right\} &= 0 \end{aligned} \qquad (8)$$

which are of course the inviscid equations with the strained coordinate s as the independent variable. To first order

$$\left(V_0^2 - \frac{2\tau_0}{5}\right)\frac{dV_1}{ds} + \frac{2}{5}V_0\frac{d\tau_1}{ds} + \left(2V_0\frac{dV_0}{ds} + \frac{2}{5}\frac{d\tau_0}{ds} - \frac{2}{5}\tau_0\right)V_1 - \frac{2}{5}\left(V_0 + \frac{dV_0}{ds}\right)\tau_1 = \frac{4}{3}V_0^2\left\{\tau_0\frac{d^2V_0}{ds^2} + \frac{dV_0}{ds}\cdot\frac{d\tau_0}{ds} - V_0\tau_0 - \frac{V_0}{2}\frac{d\tau_0}{ds}\right\} + \frac{2}{5}\tau_0V_0\frac{dX_1}{ds} \qquad (9a)$$

$$\frac{2}{3}\tau_0\frac{dV_1}{ds} + V_0\frac{d\tau_1}{ds} + \left(\frac{d\tau_0}{ds} + \frac{2}{3}\tau_0\right)V_1 + \frac{2}{3}\left(V_0 + \frac{dV_0}{ds}\right)\tau_1 = \frac{5}{3}V_0\left\{\frac{4}{3}\tau_0\left[\left(\frac{dV_0}{ds}\right)^2 - V_0\frac{dV_0}{ds} + V_0^2\right] + \tau_0\frac{d^2\tau_0}{ds^2} + \left(\frac{d\tau_0}{ds}\right)^2\right\} - \frac{2}{3}\tau_0V_0\frac{dX_1}{ds} \qquad (9b)$$

The straining of the variable x is now chosen to remove the terms in Eqs. (9) which produce the nonuniform behavior in the inviscid expansion. In Eq. (9a) it is the term $-\frac{4}{3}V_0^3\tau_0$ and in (9b) the term $\frac{20}{9}V_0^3\tau_0$. The choice

$$dX_1/ds = \tfrac{10}{3}V_0^2 \qquad (10)$$

removes both these terms. Any possible nonuniformities revealed in the higher-order analysis can be removed by appropriate choices of X_2, X_3, etc. The solution of Eqs. (8) now gives us a uniformly valid solution as $r \to \infty$, this is

$$\begin{aligned} \tau_0 &= 2(1 - V_0^2/4) \\ \rho_0 &= (4/3)^{3/2}(1 - V_0^2/4)^{3/2} \end{aligned} \qquad (11)$$

and

$$e^{-s} = (4/3)^{3/2}V_0(1 - V_0^2/4)^{3/2}$$

which are the inviscid solutions where the strained variable s has replaced x. The distribution function is given by

$$f_0 = [\rho_0/(2\pi\tau_0)^{3/2}] \exp -\{C^2/2\tau_0\} \qquad (12)$$

where, on integration of Eq. (10)

$$x = s + \frac{20}{3A}\left\{(1 - V_0^2) - 3\log\left(\frac{4 - V_0^2}{3}\right)\right\} + 0\left(\frac{1}{A^2}\right) \qquad (13)$$

X_1 vanishing when $V_0 = 1$. For $x \to \infty$, this can be inverted to give

$$s = x(1 - 40/3A) + 0(1/A^2) \qquad (14)$$

neglecting exponentially small terms in s. Finally it will be noticed that the Maxwellian distribution furnishes a uniform approximation to the full solution as $r \to \infty$ and no anisotropy in the distribution function is apparent.

References

[1] Edwards, R. H. and Cheng, H. K., "Steady Expansion of a Gas into a Vacuum," *AIAA Journal*, Vol. 4, No. 3, March 1966, pp. 558–561.

[2] Hamel, B. B. and Willis, D. R., "Kinetic Theory of Source Flow Expansions with Application to the Free Jet," *The Physics of Fluids*, Vol. 9, No. 5, May 1966, pp. 829–841.

[3] Freeman, N. C., "Expansions into Vacuum," *AIAA Journal*, Vol. 5, No. 9, Sept. 1967, pp. 1696–1698.

[4] Liepmann, H. W., Narashimha, R., and Chahine, M. T., "Structure of a Plane Shock Layer," *The Physics of Fluids*, Vol. 5, Nov. 1962, pp. 1313–1324.

A Numerical Method for the Solution of a Shell Problem

Dusan Krajcinovic*

Argonne National Laboratory, Argonne, Ill.

THE set of two partial differential equations governing the static problem of a thin, elastic, and shallow shell[1,2] is

$$K\Delta\Delta w - \Delta_k F = q, \; Eh\Delta_k w + \Delta\Delta F = 0 \qquad (1)$$

where $w(x,y)$ is the normal displacement of a point on the shell's middle surface, $F(x,y)$ stress function, E elastic modulus, h thickness of the shell, K flexural rigidity, and q the distributed load in the z direction. The symbol $\Delta\Delta$ stands for the biharmonic operator while Δ_k is defined as

$$\Delta_k(\;) = z_{,yy}(\;)_{,xx} - 2z_{,xy}(\;)_{,xy} + z_{,xx}(\;)_{,yy} \qquad (2)$$

with (,) standing for differentiation. $z(x,y)$ defines the geometry of the shell's middle surface.

In absence of concentrated edge loads the boundary conditions for the boundary simply supported along the line with

Received April 29, 1969; revision received August 19, 1969. It is my pleasant duty to acknowledge the help offered by N. Hajdin, University of Belgrade, Yugoslavia.

* Formerly at Ingersoll-Rand Research Inc., Princeton, N.J.

the normal n may be approximated by

$$w = w_{,nn} = F = F_{,nn} = 0 \tag{3}$$

The internal forces in the shell are given in terms of two unknown scalar functions w and F by

$$N_x = F_{,yy}; \quad M_x = -K(w_{,xx} + \nu w_{,yy}) \tag{4}$$

etc. With N are denoted the normal forces, whereas M stands for the bending moment. ν is the Poisson's ratio.

In general case the problem posed by (1) and (3) does not allow for a closed, analytical solution. Hence, one resorts to a numerical technique to solve a particular problem. We will present herein a numerical technique based on the method originated by N. Hajdin.[3]

Consider the shell's middle surface to be an arbitrary simply connected, convex and closed region S, bounded by a Jordan's curve ∂S. For the sake of simplicity the domain S is assumed to be symmetric with respect to one of the axes. A set of orthogonal coordinate lines $x = x_m$ $(m = 1,2, \ldots, M)$ and $y = y_n$ $(n = 1,2, \ldots, N)$ is spread over the middle surface of the shell. Coordinate lines are designed such that they intersect on the boundary ∂S (Fig. 1). The lines need not be equidistant.

Denote now the highest derivative of the function $w(x,y)$ appearing in Eq. (1) by

$$\partial^4 w/\partial x^4 = p_x(x,y) \tag{5}$$

Differential equation (5) is formally identical with the equation governing the flexure of a beam of unit rigidity, subjected to transverse loading $p_x(x,y)$.

Making use of Green's functions, i.e., solutions of the boundary value problem

$$\partial^4 a_x/\partial x^4 = \delta(x - x_0) \text{ in } S \text{ and } \partial^2 a_x/\partial x^2 = a_x = 0 \text{ on } \partial S$$

where $\delta(x - x_0)$ is the Dirac delta function, an integral equation for each $y = y_n$ is written

$$w(x,y_n) = \int_{x_1}^{x_M} a_x(x,\xi,y_n)p_x(\xi,y_n)d\xi \tag{6}$$

instead of differential equation (5). The kernel $a_x(x,\xi,y_n)$ of the integral equation is apparently the influence function for the deflection of a simply supported beam of unit flexural rigidity.

The unknown functions $p_x(x,\xi,y_n)$, comprehended as some "loads," may be approximated by a finite series

$$p_x(\xi,y_n) = \sum_{i=1}^{M} p_{in}\psi_{in}(\xi) \tag{7}$$

where p_{in} are unknown ordinates (values) of the function p_x in a number of discrete points (nodes of the adopted net), while $\psi_{in}(\xi)$ are station functions chosen to be, in our case, triangles of unit height (Fig. 1).

The displacement of an arbitrary node $x = x_m$ and $y = y_n$ may, after the substitution of (7) into (6) be written as

$$w_{mn} = w(x_m,y_n) = \sum_{i=1}^{M} p_{in}a_{im} \tag{8}$$

$$a_{im} = \int_{x_1}^{x_M} a_x(x_m,\xi,y_n)\psi_{in}(\xi)d\xi \tag{9}$$

Since $a_x(x,\xi,y_n)$ is the influence function for the deflection of a "beam" $y = y_n$, a_{im} is consequently the deflection at $x = x_m$ due to the unit triangular load $\psi_i(\xi)$. It is a matter of the elementary theory of structures to compute the displacements for all "beams" due to all elementary "loads." Let us point out at this stage that it is the number of unknown loads p_{mn}, hence functions ψ_{in}, that determines the total number of equations to be dealt with. However, to obtain better accuracy in computation of a_{im} we may compute the integral (10) in some additional intermediate points without increasing the number of equations. This is of special interest when dealing with structures of variable thickness.

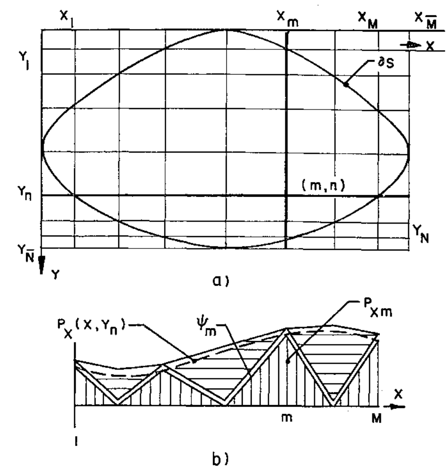


Fig. 1 a) A net of coordinate lines spread over the shell's middle surface; b) unknown loads $p_x(x,y_n)$ and adopted station functions ψ_m.

If the nodal displacements w_{mn} are comprehended as components of a vector, we write from (8) for all $y = y_n$, a matrix relation

$$\{w\} = [a_x]\{p_x\} \tag{10}$$

where the number of coordinates of vectors $\{w\}$ and $\{p\}$ equals the number of nodal points $M + N$. $[a_x]$ is the square matrix of order $M + N$.

For the beams in the y direction, starting from

$$\partial^4 w/\partial y^4 = p_y(x,y) \tag{11}$$

one derives in an analogous way

$$\{w\} = [a_y]\{p_y\} \tag{12}$$

In addition to the fourth-order derivatives, Eq. (1) contains second-order derivatives as well. Differentiating relation (6) twice with respect to x one has

$$\frac{\partial^2}{\partial x^2}[w(x,y_n)] = \int_{x_1}^{x_M} \frac{\partial^2}{\partial x^2}[a_x(x,\xi,y_n)]p_x(\xi,y_n)d\xi \tag{13}$$

or

$$\{w''\} = [a_x'']\{p_x\} \tag{14}$$

where primes denote the differentiation with respect to x. The coefficients of the matrix $[a_x'']$ are bending moments of beams $y =$ const, at point i due to the unit triangular load ψ_j, i.e.,

$$a_{im}'' = \int_{x_1}^{x_M} a_x''(x_m,y_n,\xi)\psi_{in}(\xi)d\xi \tag{15}$$

For the beams in the y direction, one analogously writes

$$\{w^{\cdot\cdot}\} = [a_y]\{p_y\} \tag{16}$$

where dots indicate differentiation with respect to y; from relations (10), (11), (14), and (16) it follows that

$$\begin{aligned}
\{w''''\} &= [a_x]^{-1}\{w\} = [d_x{}^4]\{w\} \\
\{w^{\cdot\cdot\cdot\cdot}\} &= [a_y]^{-1}\{w\} = [d_y{}^4]\{w\} \\
\{w''\} &= [a_x''][a_x]^{-1}\{w\} = [d_x{}^2]\{w\} \\
\{w^{\cdot\cdot}\} &= [a_y{}^{\cdot\cdot}][a_y]^{-1}\{w\} = [d_y{}^2]\{w\}
\end{aligned} \tag{17}$$

where

$$[d_x{}^4] = [a_x]^{-1} \text{ and } [d_x{}^2] = [a_x''][a_x]^{-1} \tag{18}$$

We finally denote the mixed fourth-order derivative from Eq. (1) by

$$\partial^4 w/\partial x^2\partial y^2 = p_{xy}(x,y) \tag{19}$$

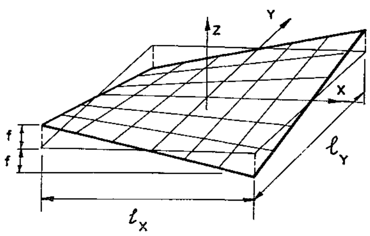


Fig. 2 A hyperbolical paraboloidal shell.

such that

$$(\partial^2/\partial y^2)[w''(x_m,y) + w_*''] = p_{xy}(x_m,y)$$

and (20)

$$(\partial^2/\partial x^2)[w^{\cdot\cdot}(x,y_n) + w^{\cdot\cdot}{}_*] = p_{xy}(x,y_n)$$

The second term in both brackets appears as a result of the fact that for nonrectangular simply supported contours $w_{,nn} = w_{,ss} = 0$, where n and s are normal and tangent to the boundary ∂S, while $w_{,xx} \neq 0$ and $w_{,yy} \neq 0$. However, from actual boundary conditions we may calculate the second-order derivatives along the boundaries. Therefore, terms with asterisks are known functions.

Differential equations (20) may now be written as integral equations

$$w''(x_m,y) = \int_{y_1}^{y_N} b_y(x_m,y,\eta)p_{xy}(x_m,\eta)d\eta - w_*''$$

$$w^{\cdot\cdot}(x,y_n) = \int_{x_1}^{x_M} b_x(x,\xi,y_n)p_{xy}(\xi,y_n)d\xi - w_*^{\cdot\cdot} \tag{21}$$

where $b_y(x_m,y,\eta)$ is the solution (Green's function) of the boundary value problem

$$\partial^2 b_y/\partial y^2 = \delta(y - y_0) \text{ in } S \text{ and } b_y = 0 \text{ on } \partial S$$

It is apparent that b_x and b_y may be computed as influence functions for the bending moments of simply supported beams, regardless of actual boundary conditions.

Taking again

$$p_{xy}(\xi,y_n) = \Sigma p_{in}\psi_{in}(\xi)$$

$$p_{xy}(x_m,\eta) = \Sigma p_{im}\psi_{im}(\eta)$$

one derives using relation (17)

$$\{w''^{\cdot\cdot}\} = [d_{xy}{}^4]\{w\} - \{w''_{0*}\} \tag{18a}$$

where

$$[d_{xy}{}^4] = [b_y]^{-1}[d_x{}^2] \text{ and } \{w''_{0*}\} = [b_y]^{-1}\{w_*''\}$$

In a similar way for the mixed second-order derivatives one has

$$\{w^{\cdot\prime}\} = [d_{xy}{}^2]\{w\} \text{ where } [d_{xy}{}^2] = [a_y{}^{\cdot\cdot\cdot}][a_x{}'''][d_{xy}{}^4]$$

where a''' and $a_y{}^{\cdot\cdot\cdot}$ are the influence functions for transverse forces of corresponding beams.

Exactly the same procedure is now repeated for the stress function F. Needless to say, in general case, the boundary conditions for F and w are not the same. Hence, the coefficients of various matrices are computed for different beams. We write

$$\{F''''\} = [d_{fx}{}^4]\{F\}, \{F''\} = [d_{fx}{}^2]\{F\} \text{ etc.} \tag{19a}$$

where for example

$$[d_{fx}{}^2] = [a''_{fx}][a_{fx}]^{-1}$$

The coefficients a''_{fij} and a_{fij} of these matrices are computed as bending moments and deflections for beams having same boundary conditions as function $F(x,y)$.

Having established matrix relations for all the necessary derivatives in (1) the differential operators $\Delta\Delta$ and Δ_k read

$$\Delta\Delta w = ([d_x{}^4] + 2[d_{xy}{}^4] + [d_y{}^4])\{w\} = [k_{11}]\{w\}$$

$$\Delta_k w = (z_{,xx}[d_y{}^2] - 2z_{,xy}[d_{xy}{}^2] + z_{,yy}[d_x{}^2])\{w\} = [k_{21}]\{w\} \tag{20a}$$

and similarly

$$\Delta\Delta F = [k_{22}]\{F\} \text{ and } \Delta_k F = [k_{12}]\{F\} \tag{21a}$$

Hence, the complete system of equations (1) is now

$$\begin{bmatrix} K\,\mathbf{k}_{11} & -\mathbf{k}_{12} \\ Eh\,\mathbf{k}_{21} & \mathbf{k}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{W} \\ \mathbf{F} \end{bmatrix} = \begin{bmatrix} \mathbf{q} \\ \mathbf{o} \end{bmatrix} \tag{22}$$

As a numerical example we consider a simply supported rectangular hyperbolic paraboloidal shell (Fig. 2) defined by $z = (4\,fxy)/(l_x l_y)$ where f is the rise of the corner and l_i lengths of supported edges.

For this case matrix, Eq. (22) may be written as

$$[\mathbf{k}_{11} + 192(f/h)^2\mathbf{k}_{12}\mathbf{k}_{22}{}^{-1}\mathbf{k}_{21}]\{w\} = \{q\} \tag{23}$$

The first part of the stiffness matrix represents the stiffness of the corresponding plate, whereas the second part reflects the additional stiffness due to the curvature of the shell. Note that both parts are positive definite matrices (second part is a congruent transformation of a positive definite matrix). It is apparent that the additional stiffness depends only on the ratio of the rise f and the thickness h of the shell.

As a numerical example we consider the shell subjected to a uniformly distributed load of $q = 0.5$ psi (0.35 t/m^2). The dimensions of the shell are $l_x = l_y = 39.37$ ft (12 m), rise $f = 23.62$ in. (60 cm), and thickness $h = 3.94$ in. (10 cm). The Poisson's ratio is taken for convenience to be zero.

Because of the symmetry of the load and geometry we consider only one quarter of the second and adopt a mesh of equidistant x and y lines as shown in Fig. 2, having only 9 nodes in observed region. The displacements w are symmetrical to both $x = 0$ and $y = 0$ lines. Therefore, coefficients a_{ij} and a_{ij}'' are sought for beams with $w = w'' = 0$ at $x = l/2$ and $w' = w''' = 0$ at $x = 0$.

On the other hand the stress function F is antisymmetric with respect to centerlines. Hence, coefficients of matrices a_{fij} and a_{fij}'' are computed for beams with $F = F'' = 0$ at $x = l/2$ and $x = 0$. The resulting bending moments M_x and shearing forces are presented in Fig. 3.

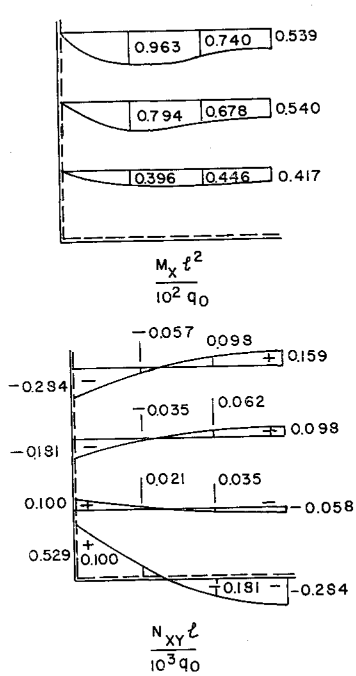


Fig. 3 Bending moments and shear forces in the shell.

In conclusion we would like to emphasize that the method itself being a numerical integration of differential equation can be applied for a variety of different problems. We note also that the basic unknown is the highest derivative appearing in the governing differential equation. Hence, the proposed numerical procedure involves numerical integration rather than numerical differentiation (as in case of the finite difference method). It is, therefore, not surprising that the obtained accuracy for the same number of equations is of superior accuracy in comparison with methods based on numerical differentiation.[4]

References

[1] Vlasov, V. Z., "General Theory of Shells and its Applications in Engineering," N-64-19883, 1964, NASA.

[2] Sanders, J. L., "Nonlinear Theories for Thin Shells," *Quarterly of Applied Mathematics*, Vol. XXI, No. 1, 1963.

[3] Hajdin, N., "A Method for Numerical Solution of Boundary Value Problems," *Translations of the Civil Engineering Department*, Vol. 4, 1958, Belgrade.

[4] Hadid, A., "An Analytical and Experimental Investigation into the Bending Theory of Elastic Conoidal Shells," Dr. Sc. dissertation, 1964, Univ. of Southampton.

Fundamental Natural Frequencies of Circular Sandwich Plates

JAO-SHIUN KAO*
Marquette University, Milwaukee, Wis.
AND
ROBERT J. ROSS†
U.S. Naval Civil Engineering Laboratory, Port Hueneme, Calif.

Introduction

VIBRATIONS of sandwich plates have been treated by many investigators. However, all of these investigations pertain to the analysis of rectangular plates; and, to our knowledge, no investigation of vibration of circular sandwich plates exists to date. A comprehensive literature survey (through 1965) may be found in Ref. 1.

The purpose of this Note is to obtain the fundamental natural frequencies of clamped and simply supported circular sandwich plates. The governing equations are developed by means of a variational theorem. For convenience of practical applications, the results are presented in a graph in terms of nondimensional parameters.

The assumptions made in the present analysis are: 1) the face members are treated as membranes; 2) the face layers may be made of different materials and unequal thickness, however the Poisson's ratio is assumed to be the same; 3) both the core and face layers are considered to be isotropic; and 4) the core is assumed to take only the transverse shear stress.

Analysis

Consider a circular sandwich plate of radius a, core thickness h_3, and two facings of thickness h_1 and h_2. Figure 1 shows a typical element of the sandwich plate. Since the fundamental frequency of vibration is the prime objective, the problem can be reduced to one of rotational symmetry.

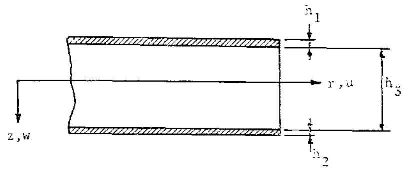
The total energy of the entire plate can be separated into the following three parts:

Received June 27, 1969; revision received August 27, 1969.

* Associate Professor of Civil Engineering.

† Civil Engineer.

Fig. 1 Cross section of plate.



1) The kinetic energy due to the deflection,

$$T = \frac{1}{2} 2\pi \int_0^a \rho w_{,t}^2 dr \tag{1}$$

where ρ is the surface density, w the deflection, and comma denotes differentiation.

2) The strain energy due to displacement of the facings,

$$U_d = \frac{1}{2} 2\pi \sum_{i=1}^{2} \frac{E_i h_i}{(1-\nu^2)} \int_0^a \left(u_{i,r}^2 + \frac{2\nu}{r} u_i u_{i,r} + \frac{1}{r} u_i^2\right) dr \tag{2}$$

where u_i is the displacement at the center of the ith face in the radial direction, and E_i and h_i are the elastic modulus and thickness of the ith facing, respectively.

3) Strain energy in the core due to shear deformation,

$$U_s = \frac{1}{2} 2\pi G h_3 \int_0^a \left[\frac{u_2 - u_1}{c} + w_{,r}\right]^2 dr \tag{3}$$

where G is the shearing modulus of the core and c is the distance between the middle planes of the facings.

The Lagrangian is then given by

$$L = T - U_d - U_s \tag{4}$$

Taking the variation of L, i.e., $\delta L = 0$, leads to the Euler equations

$$-\rho w_{,tt} + G h_3 \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\left(\frac{u_2 - u_1}{c} + w_{,r}\right) = 0 \tag{5}$$

$$-\frac{E_1 h_1}{(1-\nu^2)} \left[\frac{\nu}{r} u_{1,r} + \frac{1}{r} u_1 - \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\left(u_{1,r} + \frac{\nu}{r} u_1\right)\right] + \frac{G h_3}{c}\left(\frac{u_2 - u_1}{c} + w_{,r}\right) = 0 \tag{6}$$

$$-\frac{E_2 h_2}{(1-\nu^2)} \left[\frac{\nu}{r} u_{2,r} + \frac{1}{r} u_2 - \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\left(u_{2,r} + \frac{\nu}{r} u_2\right)\right] - \frac{G h_3}{c}\left(\frac{u_2 - u_1}{c} + w_{,r}\right) = 0 \tag{7}$$

Substituting the resulting equation obtained by differentiating Eq. (5) and dividing by Gh_3 into the difference between Eqs. (7) and (6) yields

$$\alpha = \frac{\rho D}{(G h_3)^2} w_{,ttr} - \frac{D}{G h_3} \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) w_{,r} - w_{,r} \tag{8}$$

where

$$\alpha = (u_2 - u_1)/c \tag{9a}$$

$$D = E_1 E_2 h_1 h_2 c^2/(1-\nu^2)(E_1 h_1 + E_2 h_2) \tag{9b}$$

Substituting Eq. (8) into the Euler equations reduces Eqs. (5, 6, and 7) to one equation in w,

$$\nabla^4 w = -(\rho/D) w_{,tt} + (\rho/G h_3) \nabla^2 w_{,tt} \tag{10}$$

where

$$\begin{aligned} \nabla^2 &= \partial^2/\partial r^2 + (1/r)\partial/\partial r \\ \nabla^4 &= \nabla^2 \cdot \nabla^2 \end{aligned} \tag{11}$$